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New spin Calogero–Sutherland models related to B_N-type Dunkl operators

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Abstract

We construct several new families of exactly and quasi-exactly solvable BC_N-type Calogero–Sutherland models with internal degrees of freedom. Our approach is based on the introduction of a new family of Dunkl operators of B_N type which, together with the original B_N-type Dunkl operators, are shown to preserve certain polynomial subspaces of finite dimension. We prove that a wide class of quadratic combinations involving these three sets of Dunkl operators always yields a spin Calogero–Sutherland model, which is (quasi-)exactly solvable by construction. We show that all the spin Calogero–Sutherland models obtainable within this framework can be expressed in a unified way in terms of a Weierstrass $\wp$ function with suitable half-periods. This provides a natural spin counterpart of the well-known general formula for a scalar completely integrable potential of BC_N type due to Olshanetsky and Perelomov. As an illustration of our method, we exactly compute several energy levels and their corresponding wavefunctions of an elliptic quasi-exactly solvable potential for two and three particles of spin $1/2$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The completely integrable and exactly solvable models of Calogero [1] and Sutherland [2] describe a system of N quantum particles in one dimension with long-range pairwise interaction. These models and their subsequent generalizations (see [3] and references therein for a comprehensive review) have been extensively applied in many different fields of physical interest, such as fractional statistics and anyons [4–6], quantum Hall liquids [7], Yang–Mills theories [8,9], and propagation of soliton waves [10]. A significant effort has

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been devoted over the last decade to the extension of scalar Calogero–Sutherland models to systems of particles with internal degrees of freedom or “spin” [11–20]. These models have attracted considerable interest due to their connection with integrable spin chains of Haldane–Shastry type [21,22] through the “freezing trick” of Polychronakos [23].

The exactly solvable and integrable spin models introduced in [11,14] generalize the original rational (Calogero) and trigonometric (Sutherland) scalar models, and are invariant with respect to the Weyl group of type A_N. The exact solvability of both models can be established by relating the Hamiltonian to a quadratic combination of either the Dunkl [24] or the Dunkl–Cherednik [25] operators of A_N type, whose relevance in this context was first pointed out by Polychronakos [26]. We shall use the term “Dunkl operators” to collectively refer to this type of operators. Up to the best of our knowledge, only two B_N-invariant spin Calogero–Sutherland models have been proposed so far in the literature, namely the rational and the trigonometric spin models constructed by Yamamoto in [16]. The exact solvability of the rational Yamamoto model was later proved in Ref. [19] using the Dunkl operator formalism. The exact solvability of the trigonometric Yamamoto model will be proved in this paper.

In a recent paper [20] the authors proposed a new systematic method for constructing spin Calogero–Sutherland models of type A_N. One of the key ingredients of the method was the introduction of a new family of Dunkl-type operators which, together with the Dunkl operators defined in [24,25], preserve a certain polynomial module of finite dimension. It was shown that a wide class of quadratic combinations of all three types of Dunkl operators always yields a spin Calogero–Sutherland model. In this way all the previously known exactly solvable spin Calogero–Sutherland models of A_N type are recovered and, what is more important, several new exactly and *quasi-exactly* solvable spin models are obtained. By quasi-exactly solvable (QES) we mean here that the Hamiltonian preserves a known finite-dimensional subspace of smooth functions, so that a finite subset of the spectrum can be computed algebraically; see [27–29] for further details. If the Hamiltonian leaves invariant an infinite increasing sequence of finite-dimensional subspaces, we shall say that the model is exactly solvable (ES).

In this paper we extend the method of Ref. [20] to construct new families of (Q)ES spin Calogero–Sutherland models of BC_N type. To this end, we define in Section 2 a new set of Dunkl operators of B_N type leaving invariant a certain polynomial subspace of finite dimension, which is also preserved by the original Dunkl operators of B_N type introduced in [19]. In Section 3, we show that a suitable quadratic combination of all three types of Dunkl operators discussed in Section 2 can be mapped into a multi-parameter (Q)ES physical Hamiltonian with spin. This approach is a generalization of the construction used to prove the integrability of the A_N spin Calogero–Sutherland models, in which only a *single* set of Dunkl operators is involved. Our method is also related to the so-called *hidden symmetry algebra* approach to scalar N-body QES models [30–32], where the Hamiltonian is expressed as a quadratic combination of the generators of a realization of $\mathfrak{sl}(N+1)$. We then show that the sets of Dunkl operators used in our construction are invariant under inversions and scale transformations. This property is exploited in Section 4 to perform a complete classification of the BC_N-type (Q)ES spin Calogero–Sutherland models that can be constructed with the method described in this paper. The resulting potentials can be divided into nine inequivalent classes, out of which only two (the rational and trigonometric Yamamoto models) were previously known. In particular, we obtain four new families of elliptic QES spin Calogero–Sutherland models of BC_N type. Section 5 is devoted to the discussion of the general structure of the potentials listed in Section 4. We prove that all the potentials in the classification are expressible in a unified way in terms of a Weierstrass $\wp$ function with suitable (sometimes infinite) half-periods. This provides a natural spin counterpart of Olshanetsky and Perelomov's formula for a general scalar potential related to the BC_N root system. Finally, in Section 6 we illustrate the method by exactly computing several energy levels and their corresponding eigenstates for an elliptic spin 1/2 potential in the two- and three-particle cases.

2. B_N-type Dunkl operators

In this section we introduce a new family of B_N-type Dunkl operators which will play a central role in our construction of new (Q)ES spin Calogero–Sutherland models.

Let $f(\mathbf{z})$ be an arbitrary function of $\mathbf{z} = (z_1, \dots, z_N) \in \mathbb{R}^N$. Consider the permutation operators $K_{ij} = K_{ji}$ and the sign reversing operators K_i, whose action on the function f is given by

$$(K_{ij} f)(z_1, \dots, z_i, \dots, z_j, \dots, z_N) = f(z_1, \dots, z_j, \dots, z_i, \dots, z_N),$$
$$(K_i f)(z_1, \dots, z_i, \dots, z_N) = f(z_1, \dots, -z_i, \dots, z_N), \tag{1}$$

where $i, j = 1, \dots, N$. It follows that K_{ij} and K_i verify the relations

$$K_{ij}^2 = 1, \qquad K_{ij} K_{jk} = K_{ik} K_{ij} = K_{jk} K_{ik}, \qquad K_{ij} K_{kl} = K_{kl} K_{ij},$$
$$K_i^2 = 1, \qquad K_i K_j = K_j K_i\,, \qquad K_{ij} K_k = K_k K_{ij}, \qquad K_{ij} K_j = K_i K_{ij}, \tag{2}$$

where the indices i, j, k, l take distinct values in the range $1, \dots, N$. The operators K_{ij}, K_i span the Weyl group of type B_N, also called the hyperoctahedral group. We shall also employ the customary notation $\widetilde{K}_{ij} = K_i K_j K_{ij}$. Let us consider the following set of Dunkl operators:

$$J_i^- = \frac{\partial}{\partial z_i} + a\left(\sum_{j\neq i} \frac{1}{z_i - z_j}(1 - K_{ij}) + \sum_{j\neq i} \frac{1}{z_i + z_j}(1 - \widetilde{K}_{ij})\right) + \frac{b}{z_i}(1 - K_i), \tag{3}$$

$$J_i^0 = z_i \frac{\partial}{\partial z_i} - \frac{m}{2} + \frac{a}{2}\left(\sum_{j\neq i} \frac{z_i + z_j}{z_i - z_j}(1 - K_{ij}) + \sum_{j\neq i} \frac{z_i - z_j}{z_i + z_j}(1 - \widetilde{K}_{ij})\right), \tag{4}$$

$$J_i^+ = z_i^2 \frac{\partial}{\partial z_i} - m\, z_i + a\left(\sum_{j\neq i} \frac{z_i z_j}{z_i - z_j}(1 - K_{ij}) - \sum_{j\neq i} \frac{z_i z_j}{z_i + z_j}(1 - \widetilde{K}_{ij})\right)$$
$$- b' z_i \big(1 - (-1)^m K_i\big), \tag{5}$$

where a, b, b' are nonzero real parameters, m is a nonnegative integer, and $i = 1, \dots, N$. In Eqs. (3)–(5), the symbol $\sum_{j\neq i}$ denotes summation in j with $j = 1, \dots, i-1, i+1, \dots, N$.

In general, any summation or product index without an explicit range will be understood in this paper to run from 1 to N, unless otherwise constrained. It shall also be clear in each case whether a sum symbol with more than one index present denotes single or multiple summation.

The operators J_i^- in Eq. (3) have been used by Dunkl [19] to construct a complete set of eigenvectors for Yamamoto's B_N rational spin model [16]. The operators J_i^0 were also introduced by Dunkl in Ref. [19]. To the best of our knowledge, the operators J_i^+ have not been considered previously in the literature.

The operators (3)–(5) obey the commutation relations

$$\big[J_i^{\pm}, J_j^{\pm}\big] = 0, \qquad \big[J_i^0, J_j^0\big] = \frac{a^2}{4}\sum_{k\neq i,j}(K_{ij} + \widetilde{K}_{ij})(K_{jk} + \widetilde{K}_{jk} - K_{ik} - \widetilde{K}_{ik}), \tag{6}$$

$$\big[K_{ij}, J_k^{\epsilon}\big] = 0, \qquad K_{ij}J_i^{\epsilon} = J_j^{\epsilon}K_{ij}, \qquad \big[K_i, J_j^{\epsilon}\big] = 0, \qquad K_iJ_i^{\epsilon} = (-1)^{\epsilon}J_i^{\epsilon}K_i, \tag{7}$$

where $\epsilon = \pm, 0$, and the indices i, j, k take distinct values in the range $1, \dots, N$. The operators J_i^- (respectively, J_i^+), $i = 1, \dots, N$, together with K_{ij} and K_i, $i, j = 1, \dots, N$, span a degenerate affine Hecke algebra, see [25]. The operators J_i^0 do not commute, but since $J_i^0 + \frac{a}{2}\sum_{j<i}(K_{ij} + \widetilde{K}_{ij}) - \frac{a}{2}\sum_{j>i}(K_{ij} + \widetilde{K}_{ij})$ do, it can be shown that the latter operators, together with K_{ij} and K_i, also define a degenerate affine Hecke algebra.

It is well-known [19] that the operators J_i^- and J_i^0 preserve the space $\mathcal{P}_n$ of polynomials in $z_1, \dots, z_N$ of degree at most n, for all $n \in \mathbb{N}$. Moreover, for *any* nonnegative integer n, the space $\mathcal{R}_n$ spanned by the monomials $\prod_i z_i^{l_i}$ with $0 \leqslant l_i \leqslant n$ is also invariant under the action of both J_i^- and J_i^0. Let us prove this assertion in the case of J_i^0. Since $(z_i\frac{\partial}{\partial z_i} - \frac{m}{2})\mathcal{R}_n \subset \mathcal{R}_n$, it suffices to show that

$$\frac{z_i + z_j}{z_i - z_j}(1 - K_{ij})\prod_k z_k^{l_k} \subset \mathcal{R}_n, \quad \text{and} \quad \frac{z_i - z_j}{z_i + z_j}(1 - \widetilde{K}_{ij})\prod_k z_k^{l_k} \subset \mathcal{R}_n, \tag{8}$$

for any pair of indices $1 \leqslant i \neq j \leqslant N$. For the first inclusion we note that

$$\begin{aligned}
&\frac{z_i + z_j}{z_i - z_j}(1 - K_{ij})\prod_k z_k^{l_k} \\
&\quad = \Bigg(\prod_{k\neq i,j} z_k^{l_k}\Bigg)(z_i + z_j)(z_iz_j)^{\min(l_i,l_j)}\,\mathrm{sign}(l_i - l_j)\frac{z_i^{|l_i - l_j|} - z_j^{|l_i - l_j|}}{z_i - z_j} \\
&\quad = \Bigg(\prod_{k\neq i,j} z_k^{l_k}\Bigg)(z_i + z_j)(z_iz_j)^{\min(l_i,l_j)}\,\mathrm{sign}(l_i - l_j)\sum_{k=0}^{|l_i - l_j|-1} z_i^{|l_i - l_j|-1-k}z_j^k,
\end{aligned} \tag{9}$$

where

$$\mathrm{sign}(p) = \begin{cases} -1, & p < 0, \\ 0, & p = 0, \\ 1, & p > 0. \end{cases}$$

The resulting polynomial thus belongs to $\mathcal{R}_n$. Indeed, it is a linear combination of monomials $\prod_k z_k^{\tilde{l}_k}$ with $\tilde{l}_k = l_k$ for $k \neq i, j$, and $\tilde{l}_i, \tilde{l}_j \leqslant \max(l_i, l_j)$. Likewise, the second inclusion in (8) follows from the identity

$$
\begin{aligned}
&\frac{z_i - z_j}{z_i + z_j}(1 - \widetilde{K}_{ij}) \prod_k z_k^{l_k} \\
&\quad = \left(\prod_{k \neq i,j} z_k^{l_k}\right)(z_i - z_j)(z_i z_j)^{\min(l_i,l_j)} s(l_i,l_j) \frac{z_j^{|l_i - l_j|} - (-1)^{l_i + l_j} z_i^{|l_i - l_j|}}{z_i + z_j} \\
&\quad = \left(\prod_{k \neq i,j} z_k^{l_k}\right)(z_i - z_j)(z_i z_j)^{\min(l_i,l_j)} s(l_i,l_j) \sum_{k=0}^{|l_i - l_j| - 1} (-z_i)^{|l_i - l_j| - 1 - k} z_j^k, \qquad (10)
\end{aligned}
$$

where

$$
s(p,q) = \begin{cases} 1, & p < q, \\ 0, & p = q, \\ -(-1)^{p+q}, & p > q. \end{cases}
$$

We omit the analogous proof for the operators J_i^-. Unlike the previous types of Dunkl operators, the operators J_i^+ in Eq. (5) do not preserve the polynomial spaces $\mathcal{P}_n$ and $\mathcal{R}_k$ with $k \neq m$. However, the space $\mathcal{R}_m$ is invariant under the action of J_i^+. In fact, the operator

$$
z_i^2 \frac{\partial}{\partial z_i} - m\, z_i - b' z_i \big(1 - (-1)^m K_i\big),
$$

preserves $\mathcal{R}_m$, and, just as we did in the case of J_i^0, one can show that both

$$
\frac{z_i z_j}{z_i - z_j}(1 - K_{ij}) \quad \text{and} \quad \frac{z_i z_j}{z_i + z_j}(1 - \widetilde{K}_{ij})
$$

preserve $\mathcal{R}_n$ for any nonnegative integer n.

3. BC_N-type spin many-body Hamiltonians

In Section 2 we have shown that all three sets of B_N-type Dunkl operators (3)–(5) preserve the finite-dimensional polynomial space $\mathcal{R}_m$. In this section we shall use this fundamental property to construct several families of (Q)ES many-body Hamiltonians with internal degrees of freedom.

Let $\mathfrak{S} = \operatorname{span}\{|s_1, \dots, s_N\rangle \mid s_i = -M, -M+1, \dots, M;\ M \in \frac{1}{2}\mathbb{N}\}$ be the Hilbert space of the particles' internal degrees of freedom or "spin". We shall denote by S_{ij} and S_i, $i, j = 1, \dots, N$, the spin permutation and spin reversing operators, respectively, whose action on a spin state $|s_1, \dots, s_N\rangle$ is defined by

$$
\begin{aligned}
&S_{ij}|s_1, \dots, s_i, \dots, s_j, \dots, s_N\rangle = |s_1, \dots, s_j, \dots, s_i, \dots, s_N\rangle, \\
&S_i|s_1, \dots, s_i, \dots, s_N\rangle = |s_1, \dots, -s_i, \dots, s_N\rangle. \qquad (11)
\end{aligned}
$$

The operators S_{ij} and S_i are represented in $\mathfrak{S}$ by $(2M+1)^N$-dimensional Hermitian matrices, and obey identities analogous to (2). The notation $\tilde{S}_{ij} = S_i S_j S_{ij}$ shall also be used in what follows.

We shall deal in this paper with a system of N identical fermions, so that the physical states are completely antisymmetric under permutations of the particles. A physical state ψ must therefore satisfy $\Lambda_0\psi=\psi$, where Λ_0 is the antisymmetrisation operator defined by the relations $\Lambda_0^2=\Lambda_0$ and $\Pi_{ij}\Lambda_0=-\Lambda_0$, $j>i=1,\dots,N$, with $\Pi_{ij}=K_{ij}S_{ij}$. Since $K_{ij}^2=1$, the above relations are equivalent to $K_{ij}\Lambda_0=-S_{ij}\Lambda_0$, $j>i=1,\dots,N$. For the lowest values of N, the antisymmetriser Λ_0 is given by

$$N=2:\quad \Lambda_0=\frac{1}{2}(1-\Pi_{12}),$$
$$N=3:\quad \Lambda_0=\frac{1}{6}(1-\Pi_{12}-\Pi_{13}-\Pi_{23}+\Pi_{12}\Pi_{13}+\Pi_{12}\Pi_{23}).$$

Our aim is to construct new (Q)ES Hamiltonians symmetric under the Weyl group of type B_N generated by the permutation operators Π_{ij} and the sign reversing operators K_iS_i. The corresponding algebraic eigenfunctions will be antisymmetric under a change of sign of both the spatial and spin variables of any particle, and therefore satisfy $\Lambda\psi=\psi$, where Λ is the projection on states antisymmetric under permutations and sign reversals. The total antisymmetriser Λ is determined by the relations $\Lambda^2=\Lambda$ and

$$K_{ij}\Lambda=-S_{ij}\Lambda,\qquad K_i\Lambda=-S_i\Lambda,\quad j>i=1,\dots,N. \tag{12}$$

It may be easily shown that

$$\Lambda=\frac{1}{2^N}\left(\prod_i(1-K_iS_i)\right)\Lambda_0.$$

Following closely the procedure outlined in [20], we shall consider a quadratic combination of the Dunkl operators (3)–(5) of the form

$$-H^*=\sum_i\Big(c_{++}\big(J_i^+\big)^2+c_{00}\big(J_i^0\big)^2+c_{--}\big(J_i^-\big)^2+c_0J_i^0\Big), \tag{13}$$

where $c_{++}, c_{00}, c_{--}, c_0$ are arbitrary real constants such that $c_{++}^2+c_{00}^2+c_{--}^2\neq 0$. The second-order differential-difference operator (13) possesses the following remarkable properties. First, it is a quasi-exactly solvable operator, since it leaves invariant the polynomial space $\mathcal{R}_m$. In particular, if $c_{++}=0$ the operator H^* preserves $\mathcal{R}_n$ (and $\mathcal{P}_n$) for *any* nonnegative integer n, and is therefore exactly solvable. Secondly, H^* commutes with K_{ij}, K_i, S_{ij}, and S_i for all $i,j=1,\dots,N$. This follows immediately from the commutation relations (7). Note that none of the terms

$$\sum_i\big[J_i^{\pm},J_i^0\big],\qquad \sum_i\big\{J_i^{\pm},J_i^0\big\},\qquad \sum_i J_i^{\pm}$$

commute with K_i, and for that reason they have not been included in the definition of H^*. We have also discarded the term $\sum_i\{J_i^+,J_i^-\}$ because it differs from $2\sum_i[(J_i^0)^2+(b'-b)J_i^0]$ by a constant operator.

Since H^* preserves the polynomial module $\mathcal{R}_m$, commutes with Λ, and acts trivially on $\mathfrak{S}$, the module

$$\overline{\mathcal{R}}_m=\Lambda(\mathcal{R}_m\otimes\mathfrak{S}) \tag{14}$$

is also invariant under H^*. It follows from Eqs. (12) that the action of the operators K_{ij} and K_i on the module $\overline{\mathcal{R}}_m$ coincides with that of the spin operators $-S_{ij}$ and $-S_i$, respectively. Therefore, the differential operator $\overline{H}$ obtained from H^* by the formal substitutions $K_{ij} \to -S_{ij}$, $K_i \to -S_i$, $i, j = 1, \ldots, N$, also preserves the module $\overline{\mathcal{R}}_m$. For the same reason, if the coefficient c_{++} in Eq. (13) vanishes, the operator $\overline{H}$ leaves the modules $\overline{\mathcal{R}}_n$ and $\overline{\mathcal{P}}_n = \Lambda\big(\mathcal{P}_n \otimes \mathfrak{S}\big)$ invariant for any nonnegative integer n. Using the formulae (3)–(5) in the appendix for the squares of the Dunkl operators, we get the following explicit expression for the *gauge spin Hamiltonian* $\overline{H}$:

$$
\begin{aligned}
-\overline{H} = {} & \sum_i \big(P(z_i)\partial_{z_i}^2 + Q(z_i)\partial_{z_i} + R(z_i)\big) + 4a\sum_{i\neq j}\frac{z_i\,P(z_i)}{z_i^2 - z_j^2}\partial_{z_i} \\
& - \sum_i\left(\frac{b\,c_{--}}{z_i^2}(1+S_i) + b'c_{++}z_i^2\big(1+(-1)^m S_i\big)\right) \\
& - a\sum_{i\neq j}P(z_i)\left(\frac{1+S_{ij}}{(z_i-z_j)^2} + \frac{1+\tilde S_{ij}}{(z_i+z_j)^2}\right) \\
& + \frac{ac_{++}}{2}\sum_{i\neq j}\big((z_i+z_j)^2(1+S_{ij}) + (z_i-z_j)^2(1+\tilde S_{ij})\big) + \overline{C},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
P(z) &= c_{++}z^4 + c_{00}z^2 + c_{--}, \\
Q(z) &= 2c_{++}\big(1 - m - b' + 2a(1-N)\big)z^3 \\
&\quad + \Big(c_0 + c_{00}\big(1 - m + 2a(1-N)\big)\Big)z + \frac{2bc_{--}}{z}, \\
R(z) &= c_{++}m(m-1+2b')z^2, \\
\overline{C} &= c_{00}\left[\frac{Nm^2}{4} + \frac{a^2}{12}\left(\sum_{i,j,k}{}'[4 - (S_{ij}+\tilde S_{ij})(S_{ik}+\tilde S_{ik})] + 6\sum_{i\neq j}(1 - S_iS_j)\right)\right. \\
&\quad \left. + \frac{a}{2}\sum_{i\neq j}(2 + S_{ij} + \tilde S_{ij})\right] - \frac{Nmc_0}{2}.
\end{aligned}
\tag{16}
$$

Hereafter, the symbol $\sum'_{i,j,k}$ denotes summation in i, j, k with $i \neq j \neq k \neq i$.

One of the main ingredients of our method is the fact that the gauge spin Hamiltonian $\overline{H}$ can be reduced to a *physical* spin Hamiltonian

$$
H = -\sum_i \partial_{x_i}^2 + V(\mathbf{x}),
\tag{17}
$$

where $V(\mathbf{x})$ is a Hermitian matrix-valued function, by a suitable change of variables $\mathbf{z} = \boldsymbol{\zeta}(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_N)$ and a gauge transformation with a scalar function $\mu(\mathbf{x})$, namely

$$
\mu \cdot \overline{H}|_{\mathbf{z}=\boldsymbol{\zeta}(\mathbf{x})} \cdot \mu^{-1} = H.
\tag{18}
$$

We emphasize that in general there is no (matrix or scalar) gauge factor and change of coordinates reducing a given matrix second-order differential operator in N variables to a physical Hamiltonian of the form (17); see [33,34] and references therein for more details. The quadratic combination H^* has precisely been chosen so that such a gauge factor and change of variables can be easily found for $\overline{H}$. For instance, we have omitted the otherwise valid term $\sum_i [J_i^+, J_i^-]$ because it involves first-order derivatives with matrix-valued coefficients, which are usually very difficult to gauge away. The gauge factor μ and change of variables $\mathbf{z} = \boldsymbol{\zeta}(\mathbf{x})$ in Eq. (18) are, respectively, given by

$$\mu = \exp\left(\sum_i \int^{z_i} \frac{Q(y_i)}{2P(y_i)}\,\mathrm{d}y_i\right) \prod_{i<j} \bigl(z_i^2 - z_j^2\bigr)^a \prod_i P(z_i)^{-1/4}, \tag{19}$$

and

$$x_i = \zeta^{-1}(z_i) = \int^{z_i} \frac{\mathrm{d}y}{\sqrt{P(y)}}, \quad i = 1, \dots, N. \tag{20}$$

The physical spin potential V reads

$$\begin{aligned} V = a \sum_{i \neq j} \biggl[& P(z_i) \biggl(\frac{a + S_{ij}}{(z_i - z_j)^2} + \frac{a + \tilde{S}_{ij}}{(z_i + z_j)^2} \biggr) \\ & - \frac{c_{++}}{2} \bigl((z_i + z_j)^2 (1 + S_{ij}) + (z_i - z_j)^2 (1 + \tilde{S}_{ij}) \bigr) \biggr] \\ & + \sum_i \biggl[b' c_{++}\, z_i^2 \bigl(1 + (-1)^m S_i\bigr) + \frac{b\, c_{--}}{z_i^2} (1 + S_i) + W(z_i) \biggr] + C, \end{aligned} \tag{21}$$

where

$$W(z) = \frac{1}{2}\left(Q' - \frac{P''}{2}\right) + \frac{1}{4P}\left(Q - \frac{P'}{2}\right)\left(Q - \frac{3P'}{2}\right) + c\, z^2, \tag{22}$$

and

$$\begin{aligned} C &= aN(N-1)\left[c_0 - \frac{c_{00}}{3}\bigl(a(2N-1) + 3(m-1)\bigr)\right] - \overline{C}, \\ c &= -c_{++}\bigl(2a^2(N-1)(2N-1) + 4a(N-1)(b' + m - 1) + m(2b' + m - 1)\bigr). \end{aligned} \tag{23}$$

Note that the change of variables (20), and hence the potential $V(\mathbf{x})$, are defined up to an arbitrary translation in *each* coordinate x_i, $i = 1, \dots, N$. The hermiticity of the potential (21) is a consequence of the Hermitian character of the spin operators S_{ij} and S_i.

The invariance of the module $\overline{\mathcal{R}}_m$ under the gauge spin Hamiltonian $\overline{H}$ and Eq. (18) imply that the finite-dimensional module

$$\mathcal{M}_m = \mu(\mathbf{x}) \bigl[\Lambda\bigl(\mathcal{R}_m \otimes \mathfrak{S}\bigr)\bigr]_{\mathbf{z} = \boldsymbol{\zeta}(\mathbf{x})}. \tag{24}$$

is invariant under the physical spin Hamiltonian H. Therefore, any quadratic combination H^* of the form (13) leads to a (quasi-)exactly solvable spin many-body potential (21)–(23) (provided of course that the module $\mathcal{M}_m$ is not trivial). In particular, if the coefficient c_{++} vanishes, the spin Hamiltonian H with potential (21) is exactly solvable, since it leaves invariant the infinite chains of finite-dimensional modules $\mathcal{M}_n$ and $\mathcal{N}_n = \mu(\mathbf{x})[\Lambda(\mathcal{P}_n \otimes \mathfrak{S})]_{\mathbf{z}=\boldsymbol{\zeta}(\mathbf{x})}$, $n \in \mathbb{N}$.

Our goal is to obtain a complete classification of the (Q)ES spin potentials of the form (21)–(23). The key observation used to perform this classification is the fact that different gauge spin Hamiltonians $\overline{H}$ may yield the same physical potential. This follows from the form invariance of the linear spaces $\operatorname{span}\{J_i^-(\mathbf{z}), J_i^0(\mathbf{z}), J_i^+(\mathbf{z})\}$, $i = 1, \dots, N$, under projective (gauge) and scale transformations, given respectively by

$$z_j \mapsto w_j = \frac{1}{z_j}, \qquad J_i^\epsilon(\mathbf{z}) \mapsto \widetilde{J_i^\epsilon}(\mathbf{w}) = \left(\prod_j z_j^{-m}\right) J_i^\epsilon(\mathbf{z}) \left(\prod_j z_j^m\right),$$
$$j = 1, \dots, N, \ \epsilon = \pm, 0, \tag{25}$$

and

$$z_j \mapsto w_j = \lambda z_j, \qquad J_i^\epsilon(\mathbf{z}) \mapsto \widetilde{J_i^\epsilon}(\mathbf{w}) = J_i^\epsilon(\mathbf{z}), \quad j = 1, \dots, N, \ \epsilon = \pm, 0, \tag{26}$$

where $\lambda \neq 0$ is real or purely imaginary. Indeed, we get

$$\widetilde{J_i^-}(\mathbf{w}) = -J_i^+(\mathbf{w})\big|_{b' \to b}, \qquad \widetilde{J_i^0}(\mathbf{w}) = -J_i^0(\mathbf{w}), \qquad \widetilde{J_i^+}(\mathbf{w}) = -J_i^-(\mathbf{w})\big|_{b \to b'},$$

for the projective transformations (25), and $\widetilde{J_i^\epsilon}(\mathbf{w}) = \lambda^{-\epsilon} J_i^\epsilon(\mathbf{w})$ for the scale transformations (26). This implies that the resulting quadratic combination $\widetilde{H}^*$ is still of the form (13), with (in general) different coefficients $\tilde{c}_{++}$, $\tilde{c}_{00}$, $\tilde{c}_{--}$, and $\tilde{c}_0$. Using these transformations, we can reduce the polynomial $P(z)$ in (15) to one of the following seven canonical forms:

$$\begin{aligned}
&1. \quad 1,\\
&2. \quad \pm\nu z^2,\\
&3. \quad \pm\nu\left(1+z^2\right),\\
&4. \quad \pm\nu\left(1-z^2\right)^2,\\
&5. \quad \nu\left(e^{2\mathrm{i}\theta}-z^2\right)\left(e^{-2\mathrm{i}\theta}-z^2\right),\\
&6. \quad \pm\nu\left(1-z^2\right)\left(1-k^2z^2\right),\\
&7. \quad \nu\left(1-z^2\right)\left(1-k^2+k^2z^2\right),
\end{aligned} \tag{27}$$

where $\nu > 0$, $0 < k < 1$, and $0 < \theta \leqslant \pi/4$.

4. Classification of QES spin Calogero–Sutherland models

We present in this section the complete classification of all the (Q)ES spin Calogero–Sutherland models that can be constructed applying the procedure described in the previous sections. To further simplify the classification, we note that the scaling $(c_{\epsilon\epsilon}, c_0) \mapsto (\lambda c_{\epsilon\epsilon}, \lambda c_0)$ induces the mapping

$$V(\mathbf{x}; c_{\epsilon\epsilon}, c_0) \mapsto V(\mathbf{x}; \lambda\, c_{\epsilon\epsilon}, \lambda\, c_0) = \lambda V(\sqrt{\lambda}\,\mathbf{x}; c_{\epsilon\epsilon}, c_0) \tag{28}$$

of the corresponding potentials. For this reason, in Cases 2–7 we shall only list the potential for a suitably chosen value of the parameter ν. Note furthermore that in Cases 2–4 and 6 once the potential has been computed for a positive value ν_0 of the parameter ν, its counterpart for the opposite value $\nu = -\nu_0$ can be immediately obtained using (28), namely

$$V(\mathbf{x}; -\nu_0, c_0) = -V(\mathrm{i}\mathbf{x}; \nu_0, -c_0).$$

For the models constructed to be symmetric under the Weyl group of type B_N spanned by the operators $K_{ij}S_{ij}$ and K_iS_i, $1 \leqslant i < j \leqslant N$, the change of variables $\mathbf{z} = \boldsymbol{\zeta}(\mathbf{x})$ should be an odd function of $\mathbf{x}$, since only in this case $\mathbf{x} \mapsto -\mathbf{x}$ corresponds to $\mathbf{z} \mapsto -\mathbf{z}$. In all cases except the second one, this has essentially the effect of fixing the arbitrary constants on which the change of variables (20) depends. For example, in Case 7 with $\nu = 4$ the change of variables is of the form $z_i = \pm\,\mathrm{cn}(2x_i + \xi_i \mid k) \equiv \pm\,\mathrm{cn}(2x_i + \xi_i)$, $1 \leqslant i \leqslant N$. Imposing that z_i be an odd function of x_i for all i and using the identity

$$\mathrm{cn}(2x_i + \xi_i) + \mathrm{cn}(-2x_i + \xi_i) = \frac{2\,\mathrm{cn}\,\xi_i\,\mathrm{cn}(2x_i)}{1 - k^2\,\mathrm{sn}^2\,\xi_i\,\mathrm{sn}^2(2x_i)},$$

we obtain the condition

$$\xi_i = (2l_i - 1)K, \quad l_i \in \mathbb{Z},\ 1 \leqslant i \leqslant N,$$

where $K \equiv K(k)$ is the complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

Since $\mathrm{cn}(2x_i - K + 2l_iK) = (-1)^{l_i}\,\mathrm{cn}(2x_i - K)$, symmetry under exchange of the particles requires that l_i be independent of i, so that $z_i = \pm\,\mathrm{cn}(2x_i - K)$ for all $i = 1, \ldots, N$. Taking into account that both the potential V and the gauge function μ are even functions of $\mathbf{z}$ by Eqs. (19)–(22), we see that the change of variables in this case can be taken as $z_i = \mathrm{cn}(2x_i - K)$, $1 \leqslant i \leqslant N$.

In the classification that follows, we have routinely discarded constant operators of the form

$$\begin{aligned} V_0 = \gamma_0 + \gamma_1 \sum_i S_i + \gamma_2 \sum_{i<j} S_iS_j + \gamma_3 \sum_{i<j} (S_{ij} + \tilde{S}_{ij}) \\ + \gamma_4 {\sum_{i,j,k}}' (S_{ij} + \tilde{S}_{ij})(S_{ik} + \tilde{S}_{ik}), \quad \gamma_i \in \mathbb{R}. \end{aligned} \tag{29}$$

This is justified, since the operator V_0 commutes with Λ (it actually commutes with K_{ij}, S_{ij}, K_i, and S_i for $1 \leqslant i < j \leqslant N$) and therefore preserves the spaces $\mathcal{M}_n$ and $\mathcal{N}_n$ for all n.

All the potentials in the classification presented below are singular on the hyperplanes $x_i = x_j$, $1 \leqslant i < j \leqslant N$, where they diverge as $(x_i - x_j)^{-2}$. In some cases there may be other singular hyperplanes, near which the potential behaves as the inverse squared distance to the hyperplane. We shall accordingly choose as domain of the functions in the Hilbert space of the system a maximal open subset X of the open set

$$x_N < x_{N-1} < \cdots < x_2 < x_1 \tag{30}$$

containing no singularities of the potential. In all cases except Case 2b, we shall take as boundary conditions defining the eigenfunctions of H their square integrability on the region X and their vanishing on the boundary ∂X of X faster than the square root of the distance to the boundary. Since the algebraic eigenfunctions that we shall construct are in all cases regular inside X, when this set is bounded the square integrability of the algebraic eigenfunctions on X is an automatic consequence of their vanishing on ∂X. In Case 2b, the potential is regular and periodic in each coordinate in an unbounded domain. Therefore the square integrability of the eigenfunctions should be replaced by a Bloch-type boundary condition in this case.

For each of the potentials in the classification, we shall list the domain chosen for its eigenfunctions and the restrictions imposed by the boundary conditions discussed above on the parameters on which the potential depends. In particular, the singularity of the potential at $x_i = x_j$, $1 \leqslant i < j \leqslant N$, forces the parameter a to be greater than $1/2$. Similarly, in all cases except for the second one the potential is also singular on the hyperplanes $x_i = 0$, $1 \leqslant i \leqslant N$, and the vanishing of the algebraic eigenfunctions on these hyperplanes as $|x_i|^{\frac{1}{2}+\delta}$ with $\delta > 0$ requires that $b > 1/2$. The conditions

$$a > \frac{1}{2}, \qquad b > \frac{1}{2}$$

shall therefore be understood to hold in all cases. For similar reasons, in Cases 4b, 5, and 6b we must also have

$$b' > \frac{1}{2}.$$

The potential in each case will be expressed as

$$V(\mathbf{x}) = V_{\mathrm{spin}}(\mathbf{x}) + \sum_i U(x_i),$$

where the last term, which does not contain the spin operators S_{ij} and S_i, can be viewed as the contribution of a scalar external field.

We shall use in the rest of this section the convenient abbreviations

$$x_{ij}^{\pm} = x_i \pm x_j, \qquad \alpha = a(N-1) + \frac{1}{2}(b + b' + m).$$

Case 1 $P(z) = 1$. Change of variables: $z = x$. Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \left(x_{ij}^- x_{ij}^+\right)^a \prod_i x_i^b \, \mathrm{e}^{-\frac{1}{2}\omega x_i^2}. \tag{31}$$

Scalar external potential:

$$U(x) = \omega^2 x^2. \tag{32}$$

Spin potential:

$$V_{\mathrm{spin}}(\mathbf{x}) = 2a \sum_{i<j} \left[\left(x_{ij}^-\right)^{-2}(a + S_{ij}) + \left(x_{ij}^+\right)^{-2}(a + \tilde{S}_{ij})\right] + b \sum_i x_i^{-2}(b + S_i). \tag{33}$$

Parameters: $\omega = -\frac{1}{2}c_0 > 0$. Domain: $0 < x_N < \cdots < x_1$.

Case 2a $P(z) = 4z^2$. Change of variables: $z = e^{2x}$. The most general change of variables in this case is $z_i = \lambda e^{\pm 2x_i}$, $1 \leqslant i \leqslant N$, but the following formulas are independent of the choice of sign in the exponent and the value of the constant λ. Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \big[\sinh\big(2x_{ij}^-\big)\big]^a. \tag{34}$$

Scalar external potential: $U(x) = 0$. Spin potential:

$$V_{\text{spin}}(\mathbf{x}) = 2a \sum_{i<j} \big[\sinh^{-2} x_{ij}^- (a + S_{ij}) - \cosh^{-2} x_{ij}^- (a + \tilde{S}_{ij})\big]. \tag{35}$$

Parameters: $c_0 = 4m$. Domain: $x_N < \cdots < x_1$.

Case 2b $P(z) = -4\,z^2$. Change of variables: $z = \mathrm{e}^{2\mathrm{i}x}$. Again, the most general change of variables is $z_i = \lambda e^{\pm 2\mathrm{i}x_i}$, $1 \leqslant i \leqslant N$, but the following formulas do not change when this is taken into account. Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \big[\sin\big(2x_{ij}^-\big)\big]^a. \tag{36}$$

Scalar external potential: $U(x) = 0$. Spin potential:

$$V_{\text{spin}}(\mathbf{x}) = 2a \sum_{i<j} \big[\sin^{-2} x_{ij}^- (a + S_{ij}) + \cos^{-2} x_{ij}^- (a + \tilde{S}_{ij})\big]. \tag{37}$$

Parameters: $c_0 = -4m$. Domain: $x_N < \cdots < x_1 < x_N + \pi/2$.

Both potentials in this case are invariant under a simultaneous translation of all the particles' coordinates. The choice $c_0 = \pm 4m$, which simplifies the form of the gauge factor, amounts to fixing the center of mass energy of the system. Note also that the potentials in this case do *not* possess B_N symmetry, due to the fact that the change of variables cannot be made an odd function of x for any choice of the arbitrary constants. In fact, the sign change $z_k \mapsto -z_k$ corresponds to the translation $x_k \mapsto x_k + \mathrm{i}\pi/2$ or $x_k \mapsto x_k + \pi/2$, which (as any overall translation) leaves the potential invariant. The potentials in this case are therefore best interpreted as A_N-type potentials depending both on spin permutation and sign reversing operators.

For the hyperbolic potential 2a, none of the algebraic formal eigenfunctions are true eigenfunctions, since they are not square integrable on their domain. On the other hand, the algebraic eigenfunctions of the periodic potential 2b are clearly periodic in each coordinate and regular on their domain, and thus qualify as true eigenfunctions.

Case 3a $P(z) = 4(1 + z^2)$. Change of variables: $z = \sinh(2x)$. Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \big[\sinh\big(2x_{ij}^-\big) \sinh\big(2x_{ij}^+\big)\big]^a \prod_i [\sinh(2x_i)]^b [\cosh(2x_i)]^\beta. \tag{38}$$

Scalar external potential:

$$U(x) = -4\beta(\beta - 1) \cosh^{-2}(2x). \tag{39}$$

Spin potential:

$$V_{\text{spin}}(\mathbf{x}) = 2a \sum_{i<j} \Big[\big(\sinh^{-2} x_{ij}^- - \cosh^{-2} x_{ij}^+ \big)(a + S_{ij}) \\ + \big(\sinh^{-2} x_{ij}^+ - \cosh^{-2} x_{ij}^- \big)(a + \tilde{S}_{ij}) \Big] \\ + 4b \sum_i \sinh^{-2}(2x_i)(b + S_i). \tag{40}$$

Parameters:

$$\beta = \frac{c_0}{8} - \left(a(N-1) + b + \frac{m}{2} \right) < -\big(2a(N-1) + b + m\big).$$

Domain: $0 < x_N < \cdots < x_1$.

Case 3b $P(z) = -4(1+z^2)$. Change of variables: $z = \mathrm{i}\sin(2x)$. Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \big[\sin\big(2x_{ij}^-\big) \sin\big(2x_{ij}^+\big) \big]^a \prod_i [\sin(2x_i)]^b [\cos(2x_i)]^\beta. \tag{41}$$

Scalar external potential:

$$U(x) = 4\beta(\beta - 1)\cos^{-2}(2x). \tag{42}$$

Spin potential:

$$V_{\text{spin}}(\mathbf{x}) = 2a \sum_{i<j} \Big[\big(\sin^{-2} x_{ij}^- + \cos^{-2} x_{ij}^+ \big)(a + S_{ij}) \\ + \big(\sin^{-2} x_{ij}^+ + \cos^{-2} x_{ij}^- \big)(a + \tilde{S}_{ij}) \Big] \\ + 4b \sum_i \sin^{-2}(2x_i)(b + S_i). \tag{43}$$

Parameters:

$$\beta = -\left(\frac{c_0}{8} + a(N-1) + b + \frac{m}{2} \right) > \frac{1}{2} \quad \text{or} \quad \beta = 0.$$

Domain: $0 < x_N < \cdots < x_1 < \pi/4$, if $\beta > 1/2$ and $\beta \neq 1$;
$0 < x_N < \cdots < x_1 < \frac{\pi}{2} - x_2$, if $\beta = 0, 1$.

Case 4a $P(z) = (1-z^2)^2$. Change of variables: $z = \tanh x$. Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \big(\sinh x_{ij}^- \sinh x_{ij}^+ \big)^a \prod_i \mathrm{e}^{\beta \cosh(2x_i)} (\sinh x_i)^b (\cosh x_i)^{b'+m}. \tag{44}$$

Scalar external potential:

$$U(x) = 2\beta^2 \cosh(4x) + 4\beta(1 + 2\alpha)\cosh(2x). \tag{45}$$

Spin potential:

$$V_{\text{spin}}(\mathbf{x}) = 2a \sum_{i<j} \Big[\sinh^{-2} x_{ij}^- (a + S_{ij}) + \sinh^{-2} x_{ij}^+ (a + \tilde{S}_{ij}) \Big]$$

$$+\,b\sum_i \sinh^{-2} x_i\,(b+S_i) - b'\sum_i \cosh^{-2} x_i\,\big(b'+(-1)^m S_i\big). \tag{46}$$

Parameters: $\beta=\frac{1}{8}(c_0+2(b-b'))<0$, or $\beta=0$ and $\alpha<0$.
Domain: $0<x_N<\cdots<x_1$.

Alternatively, we could have taken the change of variables as $z=\tanh(x-\frac{\mathrm{i}\pi}{2})=\coth x$. The gauge factor, external potential and spin potential become, respectively,

$$\mu(\mathbf{x})=\prod_{i<j}\big(\sinh x_{ij}^- \sinh x_{ij}^+\big)^a \prod_i \mathrm{e}^{-\beta\cosh(2x_i)}(\cosh x_i)^b(\sinh x_i)^{b'+m}, \tag{47}$$

$$U(x)=2\beta^2\cosh(4x)-4\beta(1+2\alpha)\cosh(2x), \tag{48}$$

and

$$\begin{aligned}V_{\mathrm{spin}}(\mathbf{x})&=2a\sum_{i<j}\big[\sinh^{-2}x_{ij}^-(a+S_{ij})+\sinh^{-2}x_{ij}^+(a+\tilde S_{ij})\big]\\&\quad-b\sum_i\cosh^{-2}x_i\,(b+S_i)+b'\sum_i\sinh^{-2}x_i\,\big(b'+(-1)^mS_i\big).\end{aligned} \tag{49}$$

Case 4b $P(z)=-(1-z^2)^2$. Change of variables: $z=\mathrm{i}\tan x$. Gauge factor:

$$\mu(\mathbf{x})=\prod_{i<j}\big(\sin x_{ij}^- \sin x_{ij}^+\big)^a\prod_i\mathrm{e}^{\beta\cos(2x_i)}(\sin x_i)^b(\cos x_i)^{b'+m}. \tag{50}$$

Scalar external potential:

$$U(x)=-2\beta^2\cos(4x)-4\beta(1+2\alpha)\cos(2x). \tag{51}$$

Spin potential:

$$\begin{aligned}V_{\mathrm{spin}}(\mathbf{x})&=2a\sum_{i<j}\big[\sin^{-2}x_{ij}^-(a+S_{ij})+\sin^{-2}x_{ij}^+(a+\tilde S_{ij})\big]\\&\quad+b\sum_i\sin^{-2}x_i\,(b+S_i)+b'\sum_i\cos^{-2}x_i\,\big(b'+(-1)^mS_i\big).\end{aligned} \tag{52}$$

Parameters: $\beta=-\frac{1}{8}(c_0+2(b'-b))$. Domain: $0<x_N<\cdots<x_1<\pi/2$.

The change of variable can also be taken as $z=\mathrm{i}\tan(x-\pi/2)=\mathrm{i}\cot x$. Since this is the result of applying an overall *real* translation to the particles' coordinates, we shall not list the corresponding formulas for the potential and gauge factor.

Case 5 $P(z)=(\mathrm{e}^{2\mathrm{i}\theta}-z^2)(\mathrm{e}^{-2\mathrm{i}\theta}-z^2)$. Change of variables: $z=\frac{\operatorname{sn}x\,\operatorname{dn}x}{\operatorname{cn}x}$, where the modulus of the elliptic functions is $k=\cos\theta$. We shall also use in what follows the customary notation k' for the complementary modulus $\sqrt{1-k^2}$. Gauge factor:

$$\begin{aligned}\mu(\mathbf{x})&=\prod_{i<j}\left(\frac{\operatorname{sn}x_{ij}^-\operatorname{dn}x_{ij}^-\operatorname{sn}x_{ij}^+\operatorname{dn}x_{ij}^+}{1-k^2\operatorname{sn}^2x_{ij}^-\operatorname{sn}^2x_{ij}^+}\right)^a\prod_i\exp\left\{\beta\arctan\left[\frac{k}{k'}\operatorname{cn}(2x_i)\right]\right\}\\&\quad\times[\operatorname{sn}(2x_i)]^b[1+\operatorname{cn}(2x_i)]^{\frac12(b'-b+m)}[\operatorname{dn}(2x_i)]^{-\alpha}.\end{aligned} \tag{53}$$

Scalar external potential:

$$U(x) = 4k'^2 \,\mathrm{dn}^{-2}(2x)\left[\beta^2 - \alpha(\alpha+1) - \frac{k\beta}{k'}(1+2\alpha)\,\mathrm{cn}(2x)\right]. \tag{54}$$

Spin potential:

$$\begin{aligned} V_{\mathrm{spin}}(\mathbf{x}) = 2a \sum_{i<j}\Bigg[&\left(\frac{\mathrm{dn}^2 x_{ij}^-}{\mathrm{sn}^2 x_{ij}^-} - k^2k'^2\frac{\mathrm{sn}^2 x_{ij}^+}{\mathrm{dn}^2 x_{ij}^+}\right)(a+S_{ij}) \\ &+ \left(\frac{\mathrm{dn}^2 x_{ij}^+}{\mathrm{sn}^2 x_{ij}^+} - k^2k'^2\frac{\mathrm{sn}^2 x_{ij}^-}{\mathrm{dn}^2 x_{ij}^-}\right)(a+\tilde{S}_{ij})\Bigg] \\ &+ b\sum_i \left(\frac{\mathrm{cn}\, x_i}{\mathrm{sn}\, x_i\,\mathrm{dn}\, x_i}\right)^2 (b+S_i) + b'\sum_i \left(\frac{\mathrm{sn}\, x_i\,\mathrm{dn}\, x_i}{\mathrm{cn}\, x_i}\right)^2 \big(b' + (-1)^m S_i\big). \end{aligned} \tag{55}$$

Parameters: $\beta = -\frac{1}{8kk'}(c_0 + 2(k^2-k'^2)(b-b'))$. Domain: $0 < x_N < \cdots < x_1 < K$.

An alternative form for the change of variables in this case is

$$z = \frac{\mathrm{sn}(x-K)\,\mathrm{dn}(x-K)}{\mathrm{cn}(x-K)} = -\frac{\mathrm{cn}\, x}{\mathrm{sn}\, x\,\mathrm{dn}\, x}.$$

The resulting potential is obtained from the previous one by applying the overall real translation $x_i \mapsto x_i - K$, $i = 1, \dots, N$. Note also that, although z_i is singular at the zeros of $\mathrm{cn}\, x_i$, the algebraic eigenfunctions satisfy the appropriate boundary condition on these hyperplanes on account of the inequality $b' > 1/2$ and the identity

$$1 + \mathrm{cn}(2x) = \frac{2\,\mathrm{cn}^2 x}{1 - k^2\,\mathrm{sn}^4 x}.$$

Case 6a $P(z) = 4(1-z^2)(1-k^2z^2)$. Change of variables: $z = \mathrm{sn}(2x)$. Here, as in the remaining cases, the Jacobian elliptic functions have modulus k. Gauge factor:

$$\begin{aligned} \mu(\mathbf{x}) = \prod_{i<j} &\frac{(\mathrm{sn}\, x_{ij}^-\,\mathrm{cn}\, x_{ij}^-\,\mathrm{dn}\, x_{ij}^-\,\mathrm{sn}\, x_{ij}^+\,\mathrm{cn}\, x_{ij}^+\,\mathrm{dn}\, x_{ij}^+)^a}{(1 - k^2\,\mathrm{sn}^2 x_{ij}^-\,\mathrm{sn}^2 x_{ij}^+)^{2a}} \\ &\times \prod_i [\mathrm{sn}(2x_i)]^b [\mathrm{cn}(2x_i)]^\beta [\mathrm{dn}(2x_i)]^{\beta'}. \end{aligned} \tag{56}$$

Scalar external potential:

$$U(x) = 4k'^2\big[\beta(\beta-1)\,\mathrm{cn}^{-2}(2x) - \beta'(\beta'-1)\,\mathrm{dn}^{-2}(2x)\big]. \tag{57}$$

Spin potential:

$$V_{\mathrm{spin}}(\mathbf{x}) = 2a\sum_{i<j}\Bigg[\left(\frac{\mathrm{cn}^2 x_{ij}^-\,\mathrm{dn}^2 x_{ij}^-}{\mathrm{sn}^2 x_{ij}^-} + k'^4\frac{\mathrm{sn}^2 x_{ij}^+}{\mathrm{cn}^2 x_{ij}^+\,\mathrm{dn}^2 x_{ij}^+}\right)(a+S_{ij})$$

$$+\left(\frac{\operatorname{cn}^2 x_{ij}^+ \operatorname{dn}^2 x_{ij}^+}{\operatorname{sn}^2 x_{ij}^+} + k'^4 \frac{\operatorname{sn}^2 x_{ij}^-}{\operatorname{cn}^2 x_{ij}^- \operatorname{dn}^2 x_{ij}^-}\right)(a + \tilde{S}_{ij})\Bigg]$$
$$+ 4b \sum_i \operatorname{sn}^{-2}(2x_i)(b + S_i) + 4k^2 b' \sum_i \operatorname{sn}^2(2x_i)\big(b' + (-1)^m S_i\big). \quad (58)$$

Parameters:

$$\beta = -\frac{1}{8k'^2}\big(c_0 + 4\big(1+k^2\big)(b-b')\big) - \alpha > \frac{1}{2} \quad \text{or} \quad \beta = 0,$$
$$\beta' = \frac{1}{8k'^2}\big(c_0 + 4\big(1+k^2\big)(b-b')\big) - \alpha.$$

Domain: $0 < x_N < \cdots < x_1 < K/2$, if $\beta > 1/2$ and $\beta \neq 1$;
$0 < x_N < \cdots < x_1 < K - x_2$, if $\beta = 0, 1$.

Case 6b $P(z) = -4(1-z^2)(1-k'^2 z^2)$. Change of variables:

$$z = \mathrm{i}\,\frac{\operatorname{sn}(2x)}{\operatorname{cn}(2x)}.$$

Gauge factor:

$$\mu(\mathbf{x}) = \prod_{i<j} \frac{(\operatorname{sn} x_{ij}^- \operatorname{cn} x_{ij}^- \operatorname{dn} x_{ij}^- \operatorname{sn} x_{ij}^+ \operatorname{cn} x_{ij}^+ \operatorname{dn} x_{ij}^+)^a}{(1 - k^2 \operatorname{sn}^2 x_{ij}^- \operatorname{sn}^2 x_{ij}^+)^{2a}}$$
$$\times \prod_i [\operatorname{sn}(2x_i)]^b [\operatorname{cn}(2x_i)]^{b'+m} [\operatorname{dn}(2x_i)]^{\beta'}. \quad (59)$$

Scalar external potential:

$$U(x) = 4\big[\beta(\beta-1)k^2 \operatorname{sn}^2(2x) - \beta'(\beta'-1)k'^2 \operatorname{dn}^{-2}(2x)\big]. \quad (60)$$

Spin potential:

$$V_{\text{spin}}(\mathbf{x}) = 2a \sum_{i<j} \Bigg[\left(\frac{\operatorname{dn}^2 x_{ij}^-}{\operatorname{sn}^2 x_{ij}^- \operatorname{cn}^2 x_{ij}^-} + k^4 \frac{\operatorname{sn}^2 x_{ij}^+ \operatorname{cn}^2 x_{ij}^+}{\operatorname{dn}^2 x_{ij}^+}\right)(a + S_{ij})$$
$$+\left(\frac{\operatorname{dn}^2 x_{ij}^+}{\operatorname{sn}^2 x_{ij}^+ \operatorname{cn}^2 x_{ij}^+} + k^4 \frac{\operatorname{sn}^2 x_{ij}^- \operatorname{cn}^2 x_{ij}^-}{\operatorname{dn}^2 x_{ij}^-}\right)(a + \tilde{S}_{ij})\Bigg]$$
$$+ 4b \sum_i \operatorname{sn}^{-2}(2x_i)(b + S_i) + 4k'^2 b' \sum_i \operatorname{cn}^{-2}(2x_i)\big(b' + (-1)^m S_i\big). \quad (61)$$

Parameters:

$$\beta = \frac{1}{8k^2}\big(c_0 + 4\big(1+k'^2\big)(b'-b)\big) - \alpha,$$
$$\beta' = -\frac{1}{8k^2}\big(c_0 + 4\big(1+k'^2\big)(b'-b)\big) - \alpha.$$

Domain: $0 < x_N < \cdots < x_1 < K/2$.

In spite of the singularity of z_i at the zeros of $\operatorname{cn}(2x_i)$, the vanishing of the gauge factor on these hyperplanes clearly implies that the algebraic eigenfunctions fulfill the appropriate boundary condition.

Case 7 $P(z)=4(1-z^2)(k'^2+k^2z^2)$. Change of variables:

$$z=\operatorname{cn}(2x-K)=k'\frac{\operatorname{sn}(2x)}{\operatorname{dn}(2x)}.$$

Gauge factor:

$$\mu(\mathbf{x})=\prod_{i<j}\frac{(\operatorname{sn}x_{ij}^-\operatorname{cn}x_{ij}^-\operatorname{dn}x_{ij}^-\operatorname{sn}x_{ij}^+\operatorname{cn}x_{ij}^+\operatorname{dn}x_{ij}^+)^a}{(1-k^2\operatorname{sn}^2x_{ij}^-\operatorname{sn}^2x_{ij}^+)^{2a}}\times\prod_i[\operatorname{sn}(2x_i)]^b[\operatorname{cn}(2x_i)]^\beta[\operatorname{dn}(2x_i)]^{b'+m}. \tag{62}$$

Scalar external potential:

$$U(x)=4\big[\beta'(\beta'-1)k^2\operatorname{sn}^2(2x)+\beta(\beta-1)k'^2\operatorname{cn}^{-2}(2x)\big]. \tag{63}$$

Spin potential:

$$\begin{aligned}V_{\text{spin}}(\mathbf{x})=2a\sum_{i<j}\Bigg[&\left(\frac{\operatorname{cn}^2x_{ij}^-}{\operatorname{sn}^2x_{ij}^-\operatorname{dn}^2x_{ij}^-}+\frac{\operatorname{sn}^2x_{ij}^+\operatorname{dn}^2x_{ij}^+}{\operatorname{cn}^2x_{ij}^+}\right)(a+S_{ij})\\&+\left(\frac{\operatorname{cn}^2x_{ij}^+}{\operatorname{sn}^2x_{ij}^+\operatorname{dn}^2x_{ij}^+}+\frac{\operatorname{sn}^2x_{ij}^-\operatorname{dn}^2x_{ij}^-}{\operatorname{cn}^2x_{ij}^-}\right)(a+\tilde S_{ij})\Bigg]\\&+4b\sum_i\operatorname{sn}^{-2}(2x_i)(b+S_i)-4k'^2b'\sum_i\operatorname{dn}^{-2}(2x_i)\big(b'+(-1)^mS_i\big).\end{aligned} \tag{64}$$

Parameters:

$$\beta=-\left(\frac{c_0}{8}+\frac12\big(k^2-k'^2\big)(b'-b)+\alpha\right)>\frac12\quad\text{or}\quad\beta=0,$$

$$\beta'=\frac{c_0}{8}+\frac12\big(k^2-k'^2\big)(b'-b)-\alpha.$$

Domain: $0<x_N<\cdots<x_1<K/2$, if $\beta>1/2$ and $\beta\neq1$;
$0<x_N<\cdots<x_1<K-x_2$, if $\beta=0,1$.

5. Discussion

The BC_N-type potentials constructed in the previous section can be expressed in a unified way that we shall now describe. In the first place, apart from irrelevant constant operators of the form (29), the spin potential can be written as

$$\begin{aligned}V_{\text{spin}}(\mathbf{x})=2a\sum_{i<j}\Big[&\big(v(x_{ij}^-)+v(x_{ij}^++P_1)\big)(a+S_{ij})\\&+\big(v(x_{ij}^+)+v(x_{ij}^-+P_1)\big)(a+\tilde S_{ij})\Big]\\&+b\sum_i\big(v(x_i)+v(x_i+P_1)\big)(b+S_i)\end{aligned}$$

$$+\,b'\sum_i \big(v(x_i+P_2)+v(x_i+P_1+P_2)\big)\big(b'+(-1)^m S_i\big), \tag{65}$$

where v is a (possibly degenerate) elliptic function, and P_1 and P_2 are suitably chosen primitive half-periods of v (see Table 1). In particular, when one of the periods P_i of v goes to infinity, expressions like $v(x+P_i)$ with $x \in \mathbb{R}$ finite are defined as zero. In Case 1, both periods are infinite and $v(x+P_1+P_2)$ is also defined as zero. Furthermore, the constant $K' \equiv K'(k)$ in the elliptic Cases 5–7 is the complete elliptic integral of the first kind defined by $K'(k) = K(k')$. Using this notation, it is easy to verify that in the non-degenerate elliptic Cases 5–7 the scalar external potential $U(x)$ can be written as

$$\begin{aligned} U(x) = {} & \lambda(\lambda-1)\left[v\left(x+\frac{1}{2}P_1\right)+v\left(x-\frac{1}{2}P_1\right)\right] \\ & +\lambda'(\lambda'-1)\left[v\left(x+\frac{1}{2}P_1+P_2\right)+v\left(x-\frac{1}{2}P_1+P_2\right)\right], \end{aligned} \tag{66}$$

where $\lambda = \beta$ and $\lambda' = \beta'$ in Cases 6–7, while $\lambda = -\alpha + \mathrm{i}\beta$ and $\lambda' = -\alpha - \mathrm{i}\beta$ in Case 5. Formula (66) holds also in Case 3, with $\lambda = \beta$ and $\lambda' = 0$. In Cases 1 and 4 Eq. (66) cannot be directly applied, since in these cases all the terms in (66) are either indeterminate or zero. However, the potentials in Cases 4a and 4b can be obtained from that of Case 5 in the limits $\theta \to 0$ and $\theta \to \pi/2$, respectively. Likewise, applying the rescaling $x_i \mapsto \nu x_i$ ($i = 1, \ldots, N$, $\nu > 0$) to the potential of type 3a or 3b one obtains the potential of type 1 by taking $\beta = -\omega/(4\nu^2)$ and letting $\nu \to 0$.

Table 1
Function $v(x)$ and its primitive half-periods P_i (see Eq. (65)) for each of the BC_N-type potentials in Section 4

Case	$v(x)$	P_1	P_2
1	x^{-2}	∞	∞
3a	$\sinh^{-2} x$	$\frac{\mathrm{i}\pi}{2}$	∞
3b	$\sin^{-2} x$	$\frac{\pi}{2}$	∞
4a	$\sinh^{-2} x$	∞	$\frac{\mathrm{i}\pi}{2}$
4b	$\sin^{-2} x$	∞	$\frac{\pi}{2}$
5	$\frac{\mathrm{dn}^2 x}{\mathrm{sn}^2 x}$	$K + \mathrm{i}K'$	K
6a	$\frac{\mathrm{cn}^2 x\ \mathrm{dn}^2 x}{\mathrm{sn}^2 x}$	K	$\frac{\mathrm{i}K'}{2}$
6b	$\frac{\mathrm{dn}^2 x}{\mathrm{sn}^2 x\ \mathrm{cn}^2 x}$	$\mathrm{i}K'$	$\frac{K}{2}$
7	$\frac{\mathrm{cn}^2 x}{\mathrm{sn}^2 x\ \mathrm{dn}^2 x}$	K	$\frac{1}{2}(K + \mathrm{i}K')$

The function $v(x)$ that determines the potential $V(\mathbf{x})$ in the elliptic Cases 5–7 according to Eqs. (65) and (66) can be expressed in a systematic way in terms of the Weierstrass function $\wp(x;\omega_1,\omega_3)$ with primitive half-periods $\omega_1 = K$ and $\omega_3 = \mathrm{i}K'$. Indeed, dropping inessential constant operators we have

$$v(x) = \epsilon\big[\wp(x;\omega_1,\omega_3) + \wp(x+2P_2;\omega_1,\omega_3)\big], \tag{67}$$

where P_2 is the primitive half-period of v listed in Table 1, and $\epsilon = 1$ for Cases 6–7 while $\epsilon = 1/2$ for Case 5 (the only case in which $2P_2 = 2K$ is a period of $\wp$). Since in Cases 6–7 P_1 and $2P_2$ are primitive half-periods of $\wp$, the well-known second-order modular transformation of the Weierstrass function [35] applied to Eq. (67) leads to the equality

$$v(x) = \wp(x;P_1,P_2), \tag{68}$$

where the primitive half-periods P_1 and P_2 are listed in Table 1, and we have dropped an irrelevant additive constant. Substituting Eq. (68) into Eqs. (65) and (66) and applying once again a modular transformation to the one-particle terms we readily obtain the following remarkable expression for the potential $V(\mathbf{x})$ in Cases 5–7:

$$\begin{aligned} V(\mathbf{x}) = 2a\sum_{i<j}\Big[&\big(\wp\big(x_{ij}^-;P_1,P_2\big)+\wp\big(x_{ij}^+ + P_1;P_1,P_2\big)\big)(a+S_{ij})\\ &+\big(\wp\big(x_{ij}^+;P_1,P_2\big)+\wp\big(x_{ij}^- + P_1;P_1,P_2\big)\big)(a+\tilde S_{ij})\Big]\\ &+4b\sum_i \wp(2x_i;P_1,2P_2)(b+S_i)\\ &+4b'\sum_i \wp(2x_i+2P_2;P_1,2P_2)\big(b'+(-1)^m S_i\big)\\ &+4\sum_i\big[\lambda(\lambda-1)\wp(2x_i+P_1;P_1,2P_2)\\ &+\lambda'(\lambda'-1)\wp(2x_i+P_1+2P_2;P_1,2P_2)\big]. \end{aligned} \tag{69}$$

One of the main results in this paper is thus the fact that the potential (69) is QES provided that the *ordered* pair (P_1,P_2) is chosen from Cases 5–7 in Table 1. In fact, the remaining BC_N-type (Q)ES spin potentials listed in Section 4 can be obtained from the potentials in Eqs. (69) by sending one or both of the half-periods of the Weierstrass function to infinity. This is of course reminiscent of the analogous property of the integrable scalar Calogero–Sutherland models associated to root systems [3].

The potentials in Cases 1, 2, and 3 are ES for all values of the parameters. (In Case 3, the dependence of the parameter β on m through α can be absorbed in the coefficient c_0.) The potentials of type 4 are also ES for $\beta = 0$. The elliptic potentials in Cases 5–7 are always QES.

All the potentials presented in Section 4 are new, except for Cases 1 and 4. Case 1 is the rational B_N-type model introduced by Yamamoto [16] and studied by Dunkl [19]. Case 4b for $\beta = 0$ is Yamamoto's B_N-type trigonometric potential with $\lambda_1 = -b$ (in the notation of

Ref. [16]), and either $\lambda_1' = -b' < -1/2$ for m even or $\lambda_1' = b' > 1/2$ for m odd. Our results thus establish the exact solvability of the trigonometric Yamamoto model when $|\lambda_1'| > 1/2$.

It should be noted that the method developed in this paper admits a number of straightforward generalizations. In the first place, the algebraic states could be chosen symmetric under sign reversals. The resulting Hamiltonians would coincide with the ones presented in Section 4 with S_i replaced by $-S_i$. In particular, if $b = b' = 0$ one can obtain algebraic eigenfunctions of *both* types (symmetric and antisymmetric under sign reversals) for the *same* Hamiltonian. The construction can also be applied to a system of N identical bosons, just by replacing the antisymmetriser Λ_0 by the projector on states symmetric under permutations of the particles. Choosing a system of fermions is motivated by the fact that the internal degrees of freedom can be naturally interpreted as the physical spin of the particles when $M = 1/2$.

The procedure described in Section 3 relies on the algebraic identities analogous to (2) satisfied by the spin operators S_{ij} and S_i, and not on the particular realization (11). For instance, replacing the operators S_{ij} by new operators $\hat{S}_{ij}$ spanning one of the anyon-like realizations introduced by Basu-Mallick [17] would yield further families of (Q)ES spin Calogero–Sutherland models.

6. Exact solutions for an elliptic QES model

As an illustration of the procedure described in the previous sections, we shall now compute the algebraic sector of the spectrum for the elliptic QES potential of type 6a in Eqs. (57)–(58) in the case of two and three particles of spin 1/2 ($N = 2, 3$ and $M = 1/2$), for $m = 1, 2, 3$. Note that in the spin 1/2 case, the spin permutation and sign reversing operators S_{ij} and S_i can be expressed in terms of the usual one-particle SU(2) spin operators $\boldsymbol{\sigma}_i = (\sigma_i^1, \sigma_i^2, \sigma_i^3)$ in the more familiar way

$$S_{ij} = 2\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + \frac{1}{2}, \qquad S_i = 2\sigma_i^1.$$

The operator H^* corresponding to the potential (57), (58) reads

$$H^* = -\sum_{i=1}^{N} \Big(4k^2\big(J_i^+\big)^2 - 4\big(1+k^2\big)\big(J_i^0\big)^2 + 4\big(J_i^-\big)^2 + c_0 J_i^0\Big) + \overline{C}^* + E_0, \tag{70}$$

where $\overline{C}^*$ is the constant operator obtained by replacing S_{ij} by $-K_{ij}$ and S_i by $-K_i$ in the expression (16) for $\overline{C}$, and the scalar constant E_0 is given by

$$\begin{aligned} E_0 = {} & c_0 N\left(a(N-1) + b' + m + \frac{1}{2}\right) \\ & - 2N\big(1+k^2\big)\bigg(2a(N-1)(2b'+m) + 2b'(b'+m+1) \\ & + m + \frac{2}{3}(N-1)(2N-1)a^2\bigg). \end{aligned}$$

Let us first consider the two-particle case ($N=2$), for which the spin space $\mathfrak{S}$ is spanned by the four spin states $|\pm\pm\rangle \equiv |\pm\frac{1}{2}\pm\frac{1}{2}\rangle$. For $m=1$ the polynomial module $\overline{\mathcal{R}}_1$ is the one-dimensional space span$\{\varphi_1\}$, with

$$\varphi_1 = (z_1 - z_2)(|{+}{+}\rangle - |{-}{-}\rangle) + (z_1 + z_2)(|{-}{+}\rangle - |{+}{-}\rangle). \tag{71}$$

Therefore, the spin state

$$\begin{aligned}\psi_1(\mathbf{x}) = \frac{\mu(\mathbf{x})}{1 - k^2 \operatorname{sn}^2 x_{12}^- \operatorname{sn}^2 x_{12}^+}\big[&\operatorname{sn} x_{12}^- \operatorname{cn} x_{12}^+ \operatorname{dn} x_{12}^+ (|{+}{+}\rangle - |{-}{-}\rangle)\\ &+ \operatorname{sn} x_{12}^+ \operatorname{cn} x_{12}^- \operatorname{dn} x_{12}^- (|{-}{+}\rangle - |{+}{-}\rangle)\big]\end{aligned} \tag{72}$$

is an eigenfunction of the Hamiltonian of type 6a, where the gauge factor $\mu(\mathbf{x})$ is given in Eq. (56). The corresponding eigenvalue is $E_1 = E_0 - c_0$.

If $m=2$, the antisymmetrised polynomial module $\overline{\mathcal{R}}_2$ is the three-dimensional space span$\{\varphi_1, \varphi_2, \varphi_3\}$, where φ_1 is given by Eq. (71) and

$$\begin{aligned}\varphi_2 &= \big(z_1^2 - z_2^2\big)(|{+}{+}\rangle + |{-}{-}\rangle - |{+}{-}\rangle - |{-}{+}\rangle),\\ \varphi_3 &= z_1 z_2 (z_1 - z_2)(|{+}{+}\rangle - |{-}{-}\rangle) + z_1 z_2 (z_1 + z_2)(|{+}{-}\rangle - |{-}{+}\rangle).\end{aligned} \tag{73}$$

The matrix of the gauge spin Hamiltonian $\overline{H}$ (or H^*) in the basis $\{\varphi_1, \varphi_2, \varphi_3\}$ is given by

$$\begin{pmatrix} E_0 - c_0 - 4(1+k^2) & 0 & 8(2b+1) \\ 0 & E_0 - 2c_0 & 0 \\ 8(2b'+1)k^2 & 0 & E_0 - 3c_0 - 4(1+k^2) \end{pmatrix},$$

whose eigenvalues are

$$E_{1,3} = E_0 - 2c_0 - 4\big(1+k^2\big) \mp \Delta, \qquad E_2 = E_0 - 2c_0,$$

where $\Delta = [c_0^2 + 64k^2(2b+1)(2b'+1)]^{1/2}$. The corresponding physical wavefunctions are

$$\begin{aligned}\psi_{1,3}(\mathbf{x}) = \frac{\mu(\mathbf{x})}{1 - k^2 \operatorname{sn}^2 x_{12}^- \operatorname{sn}^2 x_{12}^+}\Big[&\operatorname{sn} x_{12}^- \operatorname{cn} x_{12}^+ \operatorname{dn} x_{12}^+\\ &\times \big(c_0 \mp \Delta + 8k^2(2b'+1)\operatorname{sn}(2x_1)\operatorname{sn}(2x_2)\big)(|{+}{+}\rangle - |{-}{-}\rangle)\\ &+ \operatorname{sn} x_{12}^+ \operatorname{cn} x_{12}^- \operatorname{dn} x_{12}^-\\ &\times \big(c_0 \mp \Delta - 8k^2(2b'+1)\operatorname{sn}(2x_1)\operatorname{sn}(2x_2)\big)(|{-}{+}\rangle - |{+}{-}\rangle)\Big],\\ \psi_2(\mathbf{x}) = \mu(\mathbf{x})&\frac{\operatorname{sn} x_{12}^- \operatorname{cn} x_{12}^- \operatorname{dn} x_{12}^- \operatorname{sn} x_{12}^+ \operatorname{cn} x_{12}^+ \operatorname{dn} x_{12}^+}{(1 - k^2 \operatorname{sn}^2 x_{12}^- \operatorname{sn}^2 x_{12}^+)^2}\\ &\times (|{+}{+}\rangle + |{-}{-}\rangle - |{+}{-}\rangle - |{-}{+}\rangle).\end{aligned} \tag{74}$$

For $m=3$ the antisymmetrised polynomial module $\overline{\mathcal{R}}_3$ is spanned by the spin functions $\varphi_1, \dots, \varphi_6$, where

$$\begin{aligned}\varphi_4 &= \big(z_1^3 - z_2^3\big)(|{+}{+}\rangle - |{-}{-}\rangle) + \big(z_1^3 + z_2^3\big)(|{-}{+}\rangle - |{+}{-}\rangle),\\ \varphi_5 &= z_1 z_2 \big(z_1^2 - z_2^2\big)(|{+}{+}\rangle + |{-}{-}\rangle + |{+}{-}\rangle + |{-}{+}\rangle),\\ \varphi_6 &= z_1^2 z_2^2 (z_1 - z_2)(|{+}{+}\rangle - |{-}{-}\rangle) + z_1^2 z_2^2 (z_1 + z_2)(|{-}{+}\rangle - |{+}{-}\rangle).\end{aligned} \tag{75}$$

The matrix representing $\overline{H} - E_0$ in the basis $\{\varphi_1, \dots, \varphi_6\}$ is

$$\begin{pmatrix} -c_0 - 8(1+k^2) & 0 & 8(2b+1) & -8(4a+2b+3) & 0 & 0 \\ 0 & -2c_0 - 8(1+k^2) & 0 & 0 & 0 & 0 \\ 8(2b'+3)k^2 & 0 & -3c_0 - 16(1+k^2) & -16a(1+k^2) & 0 & 8(2b+3) \\ -8(2b'+1)k^2 & 0 & 0 & -3c_0 + 16a(1+k^2) & 0 & -8(2b+1) \\ 0 & 0 & 0 & 0 & -4c_0 - 8(1+k^2) & 0 \\ 0 & 0 & 8(2b'+1)k^2 & -8(4a+2b'+3)k^2 & 0 & -5c_0 - 8(1+k^2) \end{pmatrix}$$

Clearly, $E_0 - 2c_0 - 8(1+k^2)$ and $E_0 - 4c_0 - 8(1+k^2)$ belong to the spectrum of $\overline{H}$ and hence of H, with corresponding eigenfunctions given, respectively, by $\psi_2(\mathbf{x})$ in Eq. (74) and

$$\psi_5(\mathbf{x}) = \mu(\mathbf{x}) \frac{\operatorname{sn} x_{12}^- \operatorname{cn} x_{12}^- \operatorname{dn} x_{12}^- \operatorname{sn} x_{12}^+ \operatorname{cn} x_{12}^+ \operatorname{dn} x_{12}^+}{(1 - k^2 \operatorname{sn}^2 x_{12}^- \operatorname{sn}^2 x_{12}^+)^2} \operatorname{sn}(2x_1) \operatorname{sn}(2x_2) \\ \times (|{+}{+}\rangle + |{-}{-}\rangle + |{+}{-}\rangle + |{-}{+}\rangle). \tag{76}$$

The remaining algebraic levels are the roots of a fourth degree polynomial, whose expression is too long to display here. For instance, if $a = b = b' = 1$, $k^2 = 1/2$, and $c_0 = -14$ (so that $\beta = 0$), the algebraic levels are approximately $E_1 = -327.4$, $E_2 = -288$, $E_3 = -281.4$, $E_4 = -262.2$, $E_5 = -260$, and $E_6 = -201.0$.

In the three-particle case, the spin space $\mathfrak{S}$ is spanned by the eight states $|{\pm}{\pm}{\pm}\rangle$. If $m = 1$, the antisymmetrised space $\overline{\mathcal{R}}_1$ is trivial. For $m = 2$, the antisymmetrised space $\overline{\mathcal{R}}_2$ is spanned by the single state

$$\varphi_1 = z_{12}^- z_{13}^- z_{23}^- (|{+}{+}{+}\rangle + |{-}{-}{-}\rangle) - z_{12}^- z_{13}^+ z_{23}^+ (|{+}{+}{-}\rangle + |{-}{-}{+}\rangle) \\ - z_{12}^+ z_{13}^+ z_{23}^- (|{+}{-}{-}\rangle + |{-}{+}{+}\rangle) + z_{12}^+ z_{13}^- z_{23}^+ (|{+}{-}{+}\rangle + |{-}{+}{-}\rangle), \tag{77}$$

where $z_{ij}^\pm = z_i \pm z_j$. Consequently, $\psi_1(\mathbf{x}) = \mu(\mathbf{x}) \varphi_1(\mathbf{z})$, with $z_i = \operatorname{sn}(2x_i)$, is an eigenfunction of H with eigenvalue $E_0 - 3c_0 - 4(1+k^2)$.

When $m = 3$, a basis for $\overline{\mathcal{R}}_3$ is given by the function φ_1 in Eq. (77) and

$$\begin{aligned} \varphi_2 = {} & z_{12}^- z_{13}^- z_{23}^- (z_1 + z_2 + z_3)(|{+}{+}{+}\rangle - |{-}{-}{-}\rangle) \\ & + z_{12}^- z_{13}^+ z_{23}^+ (z_1 + z_2 - z_3)(|{-}{-}{+}\rangle - |{+}{+}{-}\rangle) \\ & + z_{12}^+ z_{13}^+ z_{23}^- (-z_1 + z_2 + z_3)(|{+}{-}{-}\rangle - |{-}{+}{+}\rangle) \\ & + z_{12}^+ z_{13}^- z_{23}^+ (z_1 - z_2 + z_3)(|{+}{-}{+}\rangle - |{-}{+}{-}\rangle), \\ \varphi_3 = {} & z_{12}^- z_{13}^- z_{23}^- (z_1 z_2 + z_1 z_3 + z_2 z_3)(|{+}{+}{+}\rangle + |{-}{-}{-}\rangle) \\ & + z_{12}^- z_{13}^+ z_{23}^+ (-z_1 z_2 + z_1 z_3 + z_2 z_3)(|{-}{-}{+}\rangle + |{+}{+}{-}\rangle) \\ & + z_{12}^+ z_{13}^+ z_{23}^- (z_1 z_2 + z_1 z_3 - z_2 z_3)(|{+}{-}{-}\rangle + |{-}{+}{+}\rangle) \\ & - z_{12}^+ z_{13}^- z_{23}^+ (z_1 z_2 - z_1 z_3 + z_2 z_3)(|{+}{-}{+}\rangle + |{-}{+}{-}\rangle), \\ \varphi_4 = {} & z_1 z_2 z_3 \big[z_{12}^- z_{13}^- z_{23}^- (|{+}{+}{+}\rangle - |{-}{-}{-}\rangle) + z_{12}^- z_{13}^+ z_{23}^+ (|{+}{+}{-}\rangle - |{-}{-}{+}\rangle) \\ & + z_{12}^+ z_{13}^+ z_{23}^- (|{-}{+}{+}\rangle - |{+}{-}{-}\rangle) + z_{12}^+ z_{13}^- z_{23}^+ (|{-}{+}{-}\rangle - |{+}{-}{+}\rangle) \big]. \end{aligned} \tag{78}$$

The matrix of $\overline{H}$ in the basis $\{\varphi_1, \dots, \varphi_4\}$ reads

$$\begin{pmatrix} E_0-3c_0-16(1+k^2) & 0 & 8(4a+2b+3) & 0 \\ 0 & E_0-4c_0+8(1+k^2)(2a-1) & 0 & 8(2b+1) \\ 8(2b'+1)k^2 & 0 & E_0-5c_0+8(1+k^2)(2a-1) & 0 \\ 0 & 8(4a+2b'+3)k^2 & 0 & E_0-6c_0-16(1+k^2) \end{pmatrix}$$

The eigenvalues of this matrix are

$$E_{1,3}=E_0-4c_0+4\bigl(1+k^2\bigr)(2a-3)\mp\Delta_-,$$
$$E_{2,4}=E_0-5c_0+4\bigl(1+k^2\bigr)(2a-3)\mp\Delta_+,$$

where

$$\Delta_+=\bigl[\bigl(c_0+4\bigl(1+k^2\bigr)(2a+1)\bigr)^2+64k^2(2b+1)(4a+2b'+3)\bigr]^{1/2},$$
$$\Delta_-=\bigl[\bigl(c_0-4\bigl(1+k^2\bigr)(2a+1)\bigr)^2+64k^2(2b'+1)(4a+2b+3)\bigr]^{1/2}.$$

The corresponding eigenfunctions are

$$\psi_{1,3}(\mathbf{x})=\mu(\mathbf{x})\bigl[\bigl(c_0-4\bigl(1+k^2\bigr)(2a+1)\mp\Delta_-\bigr)\varphi_1(\mathbf{z})+8k^2(2b'+1)\varphi_3(\mathbf{z})\bigr],$$
$$\psi_{2,4}(\mathbf{x})=\mu(\mathbf{x})\bigl[\bigl(c_0+4\bigl(1+k^2\bigr)(2a+1)\mp\Delta_+\bigr)\varphi_2(\mathbf{z})+8k^2(4a+2b'+3)\varphi_4(\mathbf{z})\bigr],$$

with $z_i=\mathrm{sn}(2x_i)$.

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Appendix A

In this appendix we present a list of identities satisfied by the Dunkl operators (3)–(5) used to compute the gauge spin Hamiltonian (15).

$$\begin{aligned}\sum_i\bigl(J_i^-\bigr)^2={}&\sum_i\partial_{z_i}^2+4a\sum_{i\neq j}\frac{z_i}{z_i^2-z_j^2}\partial_{z_i}+2b\sum_i\frac{1}{z_i}\partial_{z_i}\\&+a\sum_{i\neq j}\frac{K_{ij}-1}{(z_i-z_j)^2}+a\sum_{i\neq j}\frac{\widetilde{K}_{ij}-1}{(z_i+z_j)^2}+b\sum_i\frac{K_i-1}{z_i^2},\end{aligned}$$

$$
\begin{aligned}
\sum_i \big(J_i^0\big)^2 = {}& \sum_i \big(z_i^2\,\partial_{z_i}^2 + \big(1-m+2a(1-N)\big)z_i\,\partial_{z_i}\big) \\
&+ 4a\sum_{i\neq j}\frac{z_i^3}{z_i^2-z_j^2}\partial_{z_i} + a\sum_{i\neq j}\frac{z_i z_j}{(z_i-z_j)^2}(K_{ij}-1) \\
&- a\sum_{i\neq j}\frac{z_i z_j}{(z_i+z_j)^2}(\widetilde{K}_{ij}-1) + \frac{Nm^2}{4} \\
&+ \frac{a^2}{12}\Bigg(\sum_{i,j,k}{}'\big[4-(K_{ij}+\widetilde{K}_{ij})(K_{ik}+\widetilde{K}_{ik})\big] + 6\sum_{i\neq j}(1-K_iK_j)\Bigg), \\
\sum_i \big(J_i^+\big)^2 = {}& \sum_i \big(z_i^4\,\partial_{z_i}^2 - 2\big(b'+m-1+2a(N-1)\big)z_i^3\,\partial_{z_i}\big) \\
&+ 4a\sum_{i\neq j}\frac{z_i^5}{z_i^2-z_j^2}\partial_{z_i} + a\sum_{i\neq j}\frac{z_i^2 z_j^2}{(z_i-z_j)^2}(K_{ij}-1) \\
&+ a\sum_{i\neq j}\frac{z_i^2 z_j^2}{(z_i+z_j)^2}(\widetilde{K}_{ij}-1) \\
&+ b'\sum_i z_i^2\big((-1)^m K_i - 1\big) + m(m-1+2b')\sum_i z_i^2, \\
\sum_i J_i^0 = {}& \sum_i z_i\,\partial_{z_i} - \frac{Nm}{2}.
\end{aligned}
$$

The symbol $\sum'_{i,j,k}$ means summation in i, j, k with $i \neq j \neq k \neq i$.

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